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RELATIONS EXPRESSING HARMONIC DIVISION ON A STRAIGHT LINE.

By MR. R. D. BOHANNAN, University of Virginia.

Chasles in his *Traité de Géométrie Supérieure*, Chap. IV, gives various relations expressing harmonic division on a straight line. Most of these relations are reproduced in this paper; but they have been deduced in each case in a manner different from that followed by Chasles.

1. General truths: If a, e, a', f are four points arranged in any order on a straight line, and if α, O, are the middle points of the segments aa', ef, and m any other point on the same line, then

$$ae + ea' + a'f + fa = 0, \tag{1}$$

$$af \cdot ea' + ae \cdot a'f + aa' \cdot fe = 0, \tag{2}$$

$$ma + ma' = 2m\alpha, \tag{3}$$

$$ma \cdot ma' = \overline{m\alpha}^2 - \overline{\alpha a}^2. \tag{4}$$

2. Definition of harmonic division: $\dfrac{ae}{af} \div \dfrac{a'e}{a'f} = -1.$

$$\therefore ae \cdot a'f = -a'e \cdot af. \tag{5}$$

By (5) and (2),

$$aa' \cdot ef = 2af \cdot ea', \tag{6}$$

$$aa' \cdot fe = 2ae \cdot fa'. \tag{7}$$

These relations can also be deduced by the introduction of the point at infinity, i. For $\dfrac{i\alpha}{ia'} \div \dfrac{a\alpha}{aa'} = 2$, and $\dfrac{ia}{ia} \div \dfrac{a'a}{a'a} = 2$. These relations involve only anharmonic ratios; they are therefore true when for the harmonically conjugate points, i, α, we substitute the points e, f.

$$\therefore \frac{ef}{ea'} \div \frac{af}{aa'} = 2, \quad \frac{ef}{ea} \div \frac{a'f}{a'a} = 2,$$

or $$aa' \,.\, ef = 2af \,.\, ea', \tag{6}$$
$$aa' \,.\, fe = 2ae \,.\, fa'. \tag{7}$$

3. $$-1 = \frac{\frac{a'f}{af}}{\frac{a'e}{ae}} = \frac{\frac{a'f}{aa' \,.\, af}}{\frac{a'e}{aa' \,.\, ae}} = \frac{\frac{af - aa'}{aa' \,.\, af}}{\frac{ae - aa'}{aa' \,.\, ae}} = \frac{\frac{1}{aa'} - \frac{1}{af}}{\frac{1}{aa'} - \frac{1}{ae}};$$

$$\therefore \frac{2}{aa'} = \frac{1}{ae} + \frac{1}{af}. \tag{8}$$

4. $$mf \,.\, a'e + ma' \,.\, ef + me \,.\, fa' = 0;$$

$$\therefore 2ma' = me \,.\, \frac{2fa'}{fe} + mf \,.\, \frac{2ea'}{ef}$$

$$= me \,.\, \frac{aa'}{ae} + mf \,.\, \frac{aa'}{af}, \text{ by (6), (7);}$$

$$\therefore \frac{2ma'}{aa'} = \frac{me}{ae} + \frac{mf}{af}. \tag{9}$$

5. Multiply (6), (7):—

$$-\overline{fe}^2 \,.\, \overline{aa'}^2 = 4\, fa \,.\, fa' \,.\, ea \,.\, ea';$$

$$\therefore -\overline{fe}^2 \,.\, \overline{aa}^2 = (\overline{fa}^2 - \overline{aa}^2)(\overline{ae}^2 - \overline{aa}^2), \text{ by (4),}$$
$$= \overline{fa}^2 \,.\, \overline{ea}^2 - \overline{aa}^2(\overline{fa}^2 + \overline{ea}^2) + \overline{aa}^4.$$

But $$\overline{fe}^2 = (ae - af)^2 = \overline{ae}^2 + \overline{af}^2 - 2ae \,.\, af;$$
$$\therefore 0 = \overline{af}^2 \,.\, \overline{ae}^2 - 2ae \,.\, af \,.\, \overline{aa}^2 + \overline{aa}^4;$$
$$\therefore \overline{aa}^2 = af \,.\, ae. \tag{10}$$

Similarly, $$\overline{Oe}^2 = Oa \,.\, Oa'. \tag{11}$$

This relation may also be deduced thus:—

$$\frac{ea}{ea'} = -\frac{fa}{fa'}.$$

$$\therefore \frac{Oa - Oe}{Oa' - Oe} = -\frac{Oa - Of}{Oa' - Of} = -\frac{Oa + Oe}{Oa' + Oe}.$$

$$\therefore \overline{Oe}^2 = Oa \,.\, Oa'.$$

6. $$ea \,.\, ea' = \overline{ea}^2 - \overline{aa}^2 = \overline{ea}^2 - af \,.\, ae, \text{ by (10),}$$
$$= ea(ea + af) = ea \,.\, ef;$$

i. e. $$ea \,.\, ea' = ea \,.\, ef. \qquad (12)$$

7. By (11), $$\overline{Oe}^2 = Oa \,.\, Oa' = \overline{Oa}^2 - \overline{aa}^2, \text{ by (14)};$$

$$\therefore\ \overline{Oa}^2 - \overline{aa}^2 - \overline{Oe}^2 = 0;$$

$$\therefore\ (ma - mO^2) - \overline{aa}^2 - \overline{Oe}^2 = 0;$$

$$\therefore\ (\overline{ma}^2 - \overline{aa}^2) + (\overline{mO}^2 - \overline{Oe}^2) = 2ma \,.\, mO;$$

$$\therefore\ ma \,.\, ma' + me \,.\, mf = 2ma \,.\, mO. \qquad (13)$$

8. By (12), $$ma \,.\, ma' = ma \,.\, mm',$$
$$me \,.\, mf = mO \,.\, mm'',$$

m, a, m', a'; m, e, m'', f being harmonic ranges;

$$\therefore\ \frac{1}{ma \,.\, ma'} + \frac{1}{me \,.\, mf} = \frac{1}{mm' \,.\, mm''} \left(\frac{mm'' \,.\, mO + mm' \,.\, ma}{ma \,.\, mO} \right)$$

$$= \frac{1}{mm' \,.\, mm''} \left(\frac{me \,.\, mf + ma \,.\, ma'}{ma \,.\, mO} \right)$$

$$= \frac{2}{mm' \,.\, mm''}, \text{ by (13)};$$

i. e., $$\frac{1}{ma \,.\, ma'} + \frac{1}{me \,.\, mf} = \frac{2}{mm' \,.\, mm''}. \qquad (14)$$

9. To show, without introducing the point at infinity, that

$$\frac{ma \,.\, ma'}{na \,.\, na'} + \frac{me \,.\, mf}{ne \,.\, nf} = 2\,\frac{mn' \,.\, mn''}{nn' \,.\, nn''},$$

n being any point, and n, a, n', a'; n, e, n'', f, harmonic ranges.

$$\frac{1}{nn'} + \frac{1}{nn''} = \frac{nn' + nn''}{nn' \,.\, nn''};$$

i. e. $$\frac{na + na'}{na \,.\, na'} + \frac{ne + nf}{ne \,.\, nf} = 2\,\frac{nn' + nn''}{nn' \,.\, nn''}, \text{ by (8)};$$

$$\therefore\ \frac{mn\,(na + na') + na \,.\, na'}{na \,.\, na'} + \frac{mn\,(ne + nf) + ne \,.\, nf}{ne \,.\, nf}$$
$$= 2\,\frac{mn\,(nn' + nn'') + nn' \,.\, nn''}{nn' \,.\, nn''}.$$

Also, by (14), $$\frac{\overline{mn}^2}{na \,.\, na'} + \frac{\overline{mn}^2}{ne \,.\, nf} = 2\,\frac{\overline{mn}^2}{nn' \,.\, nn''};$$

$$\therefore \frac{\overline{mn}^2 + mn(na + na') + na \,.\, na'}{na \,.\, na'} + \frac{\overline{mn}^2 + mn\,(ne + nf) + ne \,.\, nf}{ne \,.\, nf}$$

$$= 2\,\frac{\overline{mn}^2 + mn\,(nn' + nn'') + nn' \,.\, nn''}{nn' \,.\, nn''};$$

i. e. $$\frac{(mn + na)\,(mn + na')}{na \,.\, na'} + \frac{(mn + ne)\,(mn + nf)}{ne \,.\, nf} = 2\,\frac{(mn + nn')\,(mn + nn'')}{nn' \,.\, nn''};$$

i. e. $$\frac{ma \,.\, ma'}{na \,.\, na'} + \frac{me \,.\, mf}{ne \,.\, nf} = 2\,\frac{mn' \,.\, mn''}{nn' \,.\, nn''}. \qquad (15)$$

10. $$ma \,.\, ma' + me \,.\, mf = 2\,ma \,.\, mO$$
$$= 2\,(me + ea)\,mO$$
$$= 2\,me \,.\, mO + 2ea \,.\, mO$$
$$= me\,(me + mf) + 2ea \,.\, mO$$
$$= \overline{me}^2 + me \,.\, mf + 2ea \,.\, mO$$
$$\therefore ma \,.\, ma' - \overline{me}^2 + 2ae \,.\, mO = 0. \qquad (16)$$

11. Multiply the preceding equation by ef:

$$ma \,.\, ma' \,.\, ef + \overline{me}^2 \,.\, fe - 2ae \,.\, mO \,.\, fe = 0;$$

$$\therefore ma \,.\, ma' \,.\, ef + \overline{me}^2\,(fa + ae) - ae \,.\, (me + mf)\,(me - mf) = 0;$$

$$\therefore ma \,.\, ma' \,.\, ef + \overline{me}^2 \,.\, fa + \overline{mf}^2 \,.\, ae = 0. \qquad (17)$$

12. To deduce the relation

$$\frac{ef \,.\, nn'}{na \,.\, na'} + \frac{fn'}{ne} + \frac{n'e}{nf} = 0,$$

n, a, n', a'; a, e, a', f being harmonic ranges, without introducing the point at infinity.

By (14), $$\frac{1}{na \,.\, na'} + \frac{1}{ne \,.\, nf} = \frac{2}{nn' \,.\, nn''} = \frac{2nO}{nn' \,.\, nn'' \,.\, nO} = \frac{ne + nf}{nn' \,.\, ne \,.\, nf};$$

$$\therefore \frac{nn'}{na \,.\, na'} + \frac{nn'}{ne \,.\, nf} - \frac{ne + nf}{ne \,.\, nf} = 0;$$

$$\therefore \frac{nn' \,.\, ef}{na \,.\, na'} + nn'\left(\frac{nf - ne}{ne \,.\, nf}\right) - \frac{\overline{nf}^2 - \overline{ne}^2}{ne \,.\, nf} = 0;$$

$$\therefore \frac{nn' \,.\, ef}{na \,.\, na'} + nn'\left(\frac{1}{ne} - \frac{1}{nf}\right) - \frac{nf}{ne} + \frac{ne}{nf} = 0;$$

$$\therefore \frac{nn' \cdot ef}{na \cdot na'} + \frac{1}{ne}(nn' - nf) + \frac{1}{nf}(ne - nn') = 0;$$

$$\therefore \frac{nn' \cdot ef}{na \cdot na} + \frac{fn'}{ne} + \frac{n'e}{nf} = 0. \qquad (18)$$

13. m being any point, and $a, e, a', f;\ n, a, n', a'$ harmonic ranges, to show without introducing the point at infinity, that

$$\frac{ma \cdot ma'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{me}^2}{\overline{ne}^2} \cdot fn' \cdot ne + \frac{\overline{mf}^2}{\overline{nf}^2} \cdot n'e \cdot nf = 0.$$

$$ef \cdot nn' + ne \cdot fn' + n'e \cdot nf = 0, \text{ by (2)},$$

$$\overline{mn}^2 \left\{ \frac{ef \cdot nn'}{na \cdot na} + \frac{fn'}{ne} + \frac{n'e}{nf} \right\} = 0, \text{ by (18)},$$

$$\frac{nn' \cdot (na + na')}{na \cdot na'} \cdot ef = 2\,ef = -2(fn' + n'e), \text{ by (12), (3)},$$

or
$$mn \left\{ \frac{nn'(na + na')}{na \cdot na'} \cdot ef + 2fn' + 2n'e \right\} = 0.$$

Adding these three equations, and arranging the terms,

$$\frac{\overline{mn}^2 + mn(na + na') + na \cdot na'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{mn}^2 + 2mn \cdot ne + \overline{ne}^2}{ne} \cdot fn'$$

$$+ \frac{\overline{mn}^2 + 2mn \cdot nf + \overline{nf}^2}{nf} \cdot n'e = 0.$$

i. e.,
$$\frac{(mn + na)(mn + na')}{na \cdot na'} \cdot ef \cdot nn' + \frac{(mn + ne)^2}{\overline{ne}^2} \cdot fn' \cdot ne$$

$$+ \frac{(mn + nf)^2}{\overline{nf}^2} \cdot n'e \cdot nf = 0;$$

i. e.,
$$\frac{ma \cdot ma'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{me}^2}{\overline{ne}^2} \cdot fn' \cdot ne + \frac{\overline{mf}^2}{\overline{nf}^2} \cdot n'e \cdot nf = 0. \qquad (19)$$

A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

By PROF. H. A. HOWE, Denver, Col.

The errors of logarithmic computations arise primarily from the fact that the tabular mantissas are not accurate values of the functions: though their errors are governed by laws, we may assume that if an indefinitely large number of tabular mantissas be taken, distributed uniformly throughout the table, the average of their errors, disregarding the signs, will be a 0.25 of a unit of the last decimal place. Assuming that one error is as likely to occur as another, we find that there will be as many errors above 0.25 as below it. In the discussion to follow we shall employ the following law:

The probable error of the sum of any number of quantities equals the square root of the sum of the squares of the probable errors of those quantities.

PROBLEM I.

To find the probable error of a mantissa taken from a 4, 5, or 6-place table of logarithms of natural numbers.

In the best 4, 5, or 6-place tables the printed argument contains one less figure than the corresponding function. But in computation we have, in general, as many figures in the natural numbers as in the mantissas of their logarithms. Hence in interpolation, the tabular difference is multiplied by a decimal which ranges from 0.1 to 0.9. Represent this decimal by x: let two consecutive values of tabular mantissas be a and b, $a + c$ and $b + c'$ being their true values: the apparent tabular difference is then $b - a$, its true value being $b - a + c' - c$. The interpolated mantissa (if no figures of the result are rejected) is $a + x(b - a)$. It should be $a + c + x(b - a) + x(c' - c)$.

In using an n-place table, we reject all figures beyond the n^{th}, thus introducing a probable error of 0.25 of a unit in the n^{th} place.

Therefore, the probable error of an interpolated mantissa, when carried to n places is (since c and c' have a probable ~~error~~ value of 0.25)

$$[(0.25)^2 + (0.25\,x)^2 + (0.25\,(1 - x))^2]^{\frac{1}{2}} = 0.25\,(2 - 2x + 2x^2)^{\frac{1}{2}}$$

We obtain thus a table of probable errors for different values of x.

x	Probable Error.
0.1	$\sqrt{1.82} \times 0.25 = 0.337$
0.2	$\sqrt{1.68} \times 0.25 = 0.324$
0.3	$\sqrt{1.58} \times 0.25 = 0.314$
0.4	$\sqrt{1.52} \times 0.25 = 0.308$
0.5	$\sqrt{1.50} \times 0.25 = 0.306$

0.6	$\sqrt{1.52} \times 0.25 = 0.308$
0.7	$\sqrt{1.58} \times 0.25 = 0.314$
0.8	$\sqrt{1.68} \times 0.25 = 0.324$
0.9	$\sqrt{1.82} \times 0.25 = 0.337$

But x is as likely to be 0.0 as one of the values of the preceeding table: in this case since no interpolation is performed, the probable error is 0.25. The average of these probable errors is 0.31.

I have taken six groups of ninety 4-place numbers each, and found the mantissas of their logarithms from a 4-place logarithm table. The errors of these mantissas were found by comparison with a 5-place table. The numbers were equally distributed throughout the table: care was taken to have for the final figures as many zeros as ones, twos, threes, etc. Four similar groups of ninety 5-place numbers each and one hundred numbers each of which contained six digits, were treated likewise. The average of the numerical values of the errors of the mantissas of each set was taken, and the results are given below, in units of the n^{th} place, n-place logarithms.

Sets.	Averages.
90 4-place numbers.	0.306
" "	0.302
" "	0.300
" "	0.301
" "	0.301
" "	0.291
90 5-place numbers.	0.309
" "	0.309
" "	0.327
" "	0.297
" "	0.292
100 6-place numbers.	0.299

The number of errors whose values were 0 0, 0.1, etc., are exhibited below:

Errors.	No. of Errors.	Errors.	No. of Errors.
0.0	97	0.5	93
0.1	197	0.6	53
0.2	171	0.7	46
0.3	156	0.8	28
0.4	149	0.9	10

The average of the perceding 1000 errors is

0.30

which is close to the probable error given by the theory.

PROBLEM II.

Given a 4, 5, or 6-place mantissa, to find the probable error of the value obtained for the corresponding natural number.

Let l be the given mantissa, t and t' the tabular values of the next less, and next greater mantissas, $t + c$ and $t' + c'$ being their respective real values. Then $t' - t$ is the apparent tabular difference, and $t' - t + c' - c$ the true one. To get the last figure of the natural number corresponding to the mantissa l, we divide $l - t$ by $t' - t$. Accuracy would be reached by dividing $l - (t + c)$ by $t' - t + c' - c$. Denote $t' - t$ by D, and $l - t$ by d.

$$\frac{d - c}{D + (c' - c)} = \frac{d}{D} - \frac{d}{D^2} c' - \frac{1 - \frac{d}{D}}{D} c + \frac{1 - \frac{d}{D}}{D^2} c (c' - c) + \frac{d}{D^3} c' (c' - c) + \ldots$$

In practice we employ the fraction $10 \frac{d}{D}$ to get the last figure of the natural number, but since we reject all figures of the quotation, after the first, we may be said to use $10 \frac{d}{D} \pm 0.25$, 0.25 being the probable value of the rejected figures. Therefore our error is, with sufficient approximation

$$10 \frac{d}{D^2} c' + 10 \frac{1 - \frac{d}{D}}{D} c \pm 0.25.$$

The probable values of c and c' being 0.25 we have for the probable value of this expression

$$\left[100 \frac{\left(\frac{d}{D} c'\right)^2 + \left[\left(1 - \frac{d}{D}\right) c\right]^2}{D^2} + (0.25)^2\right]^{\frac{1}{2}}$$

$$= \left(\left[\frac{10}{D} \sqrt{\left(\frac{d}{D} c'\right)^2 + \left[\left(1 - \frac{d}{D}\right) c\right]^2}\right]^2 + (0.25)^2\right)^{\frac{1}{2}}$$

$$= \left(\left[\frac{2.5}{D} \sqrt{\left(\frac{d}{D}\right)^2 + \left(1 - \frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)^{\frac{1}{2}}$$

Now $\frac{d}{D}$ ranges from 0.1 to 0.9. When $\frac{d}{D} = 0.1$, 0.2, etc. we get for $\sqrt{\frac{d^2}{D^2} + \left(1 - \frac{d}{D}\right)^2}$ the successive results 0.906, 0.825, 0.762, 0.721, 0.707, 0.721, 0.762, 0.825, and 0.906, whose average is 0.793. Hence, with sufficient accuracy, the formula for the average of the probable errors becomes

$$\left[\left(\frac{1.98}{D}\right)^2 + (0.25)^2\right]^{\frac{1}{2}}$$

If one is computing in any part of the table, he can obtain the average probable error quite closely by substituting for D the arithmetical mean of the tabular differences employed. If the *numbers* corresponding to the given mantissas be uniformly distributed throughout the table, the average of all the values of the term $\frac{1.98}{D}$ will be $\frac{1.98}{8}$ or 0.25. Substituting this in the formula we get 0.35 for the average probable error. But if the mantissas be uniformly distributed, the average value of $\frac{1.98}{D}$ is 0.18; this substituted in the formula gives 0.31.

The foregoing formula takes no account of the fact that the given mantissa is often found printed in the table. In that case the probable error would be $\frac{2.5}{D}$ in units of the n^{th} place.

I have taken a set of ninety uniformly distributed 4-place mantissas and found the corresponding four-place numbers. The average of the errors was 0.28. It happened that fifteen of the chosen mantissas were given in the table a much larger proportion than would be expected. These tended to diminish the average error.

PROBLEM III.

To adapt Problems I *and* II *to a 7-place logarithm table, or a table in which the argument has two less figures than the function.*

The discussion of Problem I would be changed by using 0.01, 0.02, . . . 0.99 for the values of x, and in Problem II these would be substituted for $\frac{d}{D}$. It is evident that the numerical results obtained would not be materially altered.

The reasoning of the two preceding problems will apply to any tables in which the successive tabular differences are so nearly equal that second differences may be neglected.

PROBLEM IV.

Given an n-place natural number: to find the maximum error of an n-place interpolated mantissa.

The expression $xc' + (1 - x)c$ was found in Problem I, and this is the actual error of a mantissa interpolated from an n-place table if the mantissa be carried beyond the n^{th} place. If it be cut off at the n^{th} place an error which may amount to 0.5 of a unit of the n^{th} place may be committed. That the expression may be a maximum, both c and c' must be maxima and must have the same sign. Making the signs the same throughout, we have

$$\text{Maximum error} = 0.5\,x + 0.5\,(1 - x) + 0.5$$
$$= 1.0.$$

This error was not reached in any one of the thousand examples mentioned under Problem I.

PROBLEM V.

Given an n-place mantissa to find the maximum error of the corresponding natural number.

Neglecting the terms which involve $c' - c$ in the first equation of Problem II, multiplying the terms by 10^f, and supposing that the natural number is carried beyond the n^{th} place, we have

$$\text{Error} = \left[\frac{d}{D^2}c' + \frac{1 - \frac{d}{D}}{D}c\right]10^f,$$

where f is the difference between the number of figures in the argument and those in the function. To have a maximum error for any given values of d and D, c and c' must be maxima and have the same sign. Then,

$$\text{Maximum error} = \frac{0.50}{D}10^f.$$

When f is unity, as in the ordinary 4, 5, and 6-place tables, we have the error equal to $\frac{5}{D}$. In these tables the smallest value of D compatible with the condition that $c = c'$ is 5. But since an error of 0.5 may be introduced, when all figures beyond tne n^{th} are rejected, we have in this case

$$\text{Maximum error} = 1.50.$$

In an ordinary 7-place table $f = 2$ and the smallest value of D compatible with the condition that $c = c'$ is 44. Hence in this case the maximum error is

$$\frac{50}{44} + 0.5 = 1.64.$$

THE KINEMATICAL METHOD OF TANGENTS.

By Prof. Wm. Woolsey Johnson, Annapolis, Md.

1. If a plane B slide in any manner upon a plane A, every point in B describes a locus in the plane A, and every point in A describes a locus in the plane B. It is immaterial which we regard as the fixed and which the moving plane; and the relative motion of the planes is obviously determined when the loci in the other plane of two points, either both in one plane or one in each plane, are given; with the single exception of the case when the loci, one in each plane, are circles of which the given points are the centres. For example, in the "tram motion" two points, P and Q, of the moving plane B describe perpendicular straight lines in A, and this defines a motion in which the middle point O of PQ describes a circle in A whose centre is at O' the intersection of the straight lines, while all points of B situated on the circumference of the circle whose diameter is PQ describe straight lines in A intersecting in the same point O'. Thus the same motion is determined when two points of the plane B describe any two intersecting straight lines, and also when two points describe a circle and one of its diameters. The reciprocal motion of A upon B is that in which two straight lines of the plane now regarded as the moving plane, slide on pins in the fixed planes, the intersection O' of the lines evidently describing a circle passing through the pins; so that the motion may also be defined as that in which a given point O' describes a circle, and a straight line of the moving plane drawn through O' slides upon a fixed point in the circumference of the circle. This is the motion by which the limaçon, or protraction of a circle from a point in its circumference, is described by any point in the moving straight line. Moreover, since the straight line joining any point of the plane A (now the moving plane) to O' passes through a fixed point in the circumference of the circle in B, every point in A describes a limaçon in B, while, as is well known, every point in B describes an ellipse in A.

2. More generally, whatever be the character of the motion, any curve in the moving plane B has an envelope in the plane A, which it touches in one or more points. Confining our attention to a single point of contact, we may consider the condition that a given curve in B shall touch a given curve in A, and it is evident that in general two such conditions determine the relative motion of A and B. When one of the pair of touching curves reduces to a point, the condition becomes that considered above in which a given point in one plane describes a given curve in the other.

When two curves touch one another, a curve parallel to one of them and at a distance c from it necessarily touches a parallel to the other curve at a distance c from it; thus the condition that the parallel shall form a touching pair is equivalent to the condition that the original curves shall touch. For example, the

condition that a circle in B shall touch a given curve in A is equivalent to the condition that a concentric circle in B shall touch one of the parallels to the curve in A, and in particular to the condition that the centre of the circle in B shall describe a certain one of these parallels in the plane A. Two equivalent conditions of course fail to determine the relative motion of the planes, as in the exceptional case considered in Art. 1.

3. The simplest kind of relative motion is that of simple rotation in which both the curves of a touching pair reduce to points; in this case each of the equivalent conditions consists in the coincidence of equal circles, sliding one upon the other, and forming the "turning pair" of the Kinematics of Machinery. This is the only case in which a single pairing determines the motion; but, as a special case, we have that in which the centre is at an infinite distance, and the sliding circles become straight lines forming the "sliding pair."

4. There is at every instant of the motion an instantaneous centre of rotation; that is to say, either plane being regarded as fixed, a point in it about which the other plane is at the instant rotating. The point of the moving plane which is at the instantaneous centre has of course no motion at the instant, but, except in the case of simple rotation considered above, the instantaneous centre shifts its position continually in each plane.

5. If from the instantaneous centre we draw a normal to any curve in the moving plane B, it is evident that the foot of the normal is the point of contact of the curve in B with its envelope in A. Thus the common normal at the point of contact of a pair of touching curves passes through the instantaneous centre; and, if in a given relative position of the planes we draw the normals corresponding to two pairs of touching curves, we shall generally determine the instantaneous centre. If, however, the touching pairs are the equivalent pairs of Art. 2, the two normals are coincident and fail to determine the point. If one of a touching pair reduces to a point, as is frequently the case with the pairs used to define the motion, the normal is determined by the locus of the point. Having determined the instantaneous centre, the normal drawn to any curve in the moving plane B determines its point of contact with its envelope in the plane A, and is also normal to that envelope; and, in particular, the line joining the instantaneous centre to any point of B is the normal to the locus of this point in the fixed plane A.

6. Tangents to a considerable number of ordinary curves may thus be determined. For example, the process when applied to the ellipse regarded as generated by the "tram motion" described in Art. 1 is as follows: Inscribe between the given point and the major axis a line equal to the minor semi-axis, produce the line to meet the minor axis, draw perpendiculars to the axes from the points thus determined; then the intersection of these perpendiculars is a

point on the normal to the ellipse at the given point. For the conchoid the process is as follows: Draw a radius vector through the given point to meet the directrix; the intersection of a perpendicular to the directrix at this point with a perpendicular to the radius vector at the pole determines the instantaneous centre and the normal at the given point. A similar construction applies to all curves defined as protractions of a given curve. In the case of the limaçon and cardioid the instantaneous centre is the point of the circular directrix opposite to that in which it is cut by the radius vector.

Again, in the three-bar motion, the instantaneous centre for the motion of the middle bar is the intersection of the other two bars. This point determines the normal to the locus of every point on or rigidly connected with the middle bar. In the motion of a crank and connecting-rod, the instantaneous centre is the intersection of the crank with a perpendicular through the piston end of the connecting-rod to the line of its motion, and this point determines the normals to the ovals described by points in the connecting-rod.

7. The loci of the instantaneous centre in the two planes are called the *centroids* of the relative motion. Since the points of the two planes which at any instant coincide with the instantaneous centre are relatively at rest, it is evident that the instantaneous centre moves at the same rate in each plane; thus the centroids roll one upon the other, the point of contact at any instant being the instantaneous centre.

8. A single pair of touching curves together with the law governing the rates at which the point of contact travels on each curve would determine the relative motion of the planes. Two particular cases may be noticed. First, that in which the point of contact travels at the same rate in each curve, in which case the curves are the centroids. Second, that in which the rate is zero in one curve; thus let a curve in the plane B remain always in contact with a given curve in A, the point of contact being a fixed point of B. The common normal is now a fixed line of B, and its envelope in A is the evolute of the given curve. Thus the fixed line in B and this evolute are the centroids of the motion, and the instantaneous centre is the centre of curvature of the given curve in A. For example, the tractrix is described by one extremity of a straight line to which it is always tangent, while the other extremity describes the directrix, and a perpendicular to the directrix at this point meets the corresponding normal to the tractrix in the centre of curvature.

[The consideration of the instantaneous centre and the beginning of kinematical methods in Geometry date from Descartes who applied the process in his study of the cycloid (*Lettres* (Aug. 23, 1638) VII, 89. Ed. Cousin). The discovery of the existence of an instantaneous centre for any motion of a plane system is due to John Bernoulli (*Opera* 4, 265). Poinsot introduced the use of cen-

troids and axoids (*Liouville's Journal*, XVI, 9–129; 286–336). The application of these methods to the study of Mechanisms was first made systematically by Willis (1841); his processes were improved by Rankine (*Machinery and Millwork*, 1871); and the whole subject was reformed and freshly stated by Reuleaux (*Kinematik*, 1874), and has since been still further developed by Grashof and others. Peculiar interest attaches to the "tram-motion" from its furnishing to Leonardo da Vinci his famous discovery of the elliptic chuck for turning ovals on the lathe (Chasles, *Aperçu* 531).—*W. M. T.*]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.*

By Mr. Marcus Baker, Washington, D. C.

In April, 1883, Mr. James Main, formerly of the U. S. Coast and Geodetic Survey, published in the *Mathematical Magazine* a collection of forty-six (46) expressions for the area of a plane triangle, prefacing it with the remark that this collection "may be regarded as a matter of curiosity," and that about one-half of the formulæ are well known.

In the following August M. Ed. Lucas reprinted this collection in *Mathesis* in a classified form, separating the formulæ into five groups and adding one formula not contained in Mr. Main's list. The collection has also been reprinted in the third number of the *Tidsskrift for Mathematik*, 1883. Some two or three additional formulæ have since been printed in various mathematical publications.

The terms in which the area is expressed in Mr. Main's collection are angles, sides, perpendiculars, and radii of inscribed, escribed, and circumscribed circles. No formulæ are given involving medians or bisectors. In numbering Mr. Main has not counted those formulæ as distinct which arise from merely permuting the letters, nor has he in *every* case given all the forms possible to be obtained by permuting the letters, though he has generally done so. As numbered, then, he counts forty-six formulæ, but if every form be counted as a distinct one the total number is ninety-four.

M. Lucas, by making all possible permutations and adding one new form, makes the number 139, to which some two or three have been added since.

As the matter has proved of interest, the following collection has been made, which is a still further extension; the additional formulæ being chiefly due to introducing the medians and bisectors, not used in the former collections. In this collection Mr. Main's mode of numbering has been followed and formulæ

*Read before the Mathematical section of the Philosophical Society of Washington, January 7, 1885.

derived by permutation are not enumerated as distinct formulæ. Moreover, formulæ expressed in the same terms, but in different form, are also considered as but one. For example, in the former lists we find

$$\Delta = 2R^2 \sin A \sin B \sin C,$$

$$\Delta = \tfrac{1}{2}R^2 (\sin 2A + \sin 2B + \sin 2C),$$

and $$\Delta = \tfrac{2}{3}R^2 [\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B)]$$

given as three distinct formulæ whereas they are here counted as *one*, and the principle involved herein is employed throughout.

The total number of formulæ for the area of a plane triangle in this collection is ninety-three, not counting those arising from permutation. If these be counted as distinct the total number is two hundred and sixty-nine.

Owing to pressure of other duties and consequent lack of time some of the groups in this collection are not so fully worked up as had been planned.

It may be noted that a number of curious and interesting theorems may be obtained by equating different expressions for the area and reducing. In this collection we have classified, for convenience, all expressions for the area into five groups and each group into two parts.

Group I contains formulæ which M. Lucas has called *unique*, i. e. formulæ which do not admit of other similar formulæ by merely permuting the letters. In such formulæ *all* the sides, *all* the perpendiculars, *all* the medians, etc. must enter if one enters.

Group II contains formulæ which admit of *two* similar expressions by permutation giving three of a kind.

Group III contains formulæ which admit of *three* similar expressions by permutation giving four of a kind.

Group IV contains formulæ which admit of *five* similar expressions by permutation giving six of a kind.

Group V contains formulæ which admit of *eleven* similar expressions by permutation giving twelve of a kind.

Each group is divided into *two* parts, the *first containing* and the *second not containing* trigonometrical functions. In addition to the foregoing a group of miscellaneous expressions, not falling within the classification used, has been added, and called the Miscellaneous Group.

The notation used is as follows:

Δ = the area of the triangle;
A, B, C = the angles;
a, b, c = the sides opposite A, B, and C respectively
s = semi perimeter = $\frac{1}{2}(a + b + c)$;

R, r, r_a, r_b, r_c = the radii of circumscribed, inscribed, escribed circles respectively;
h_a, h_b, h_c = the perpendiculars from A, B, and C respectively;
m_a, m_b, m_c = the medians from A, B, and C respectively;
$\mathfrak{z}_a$, $\mathfrak{z}_b$, $\mathfrak{z}_c$ = the bisectors (internal) of the angles A, B, and C respectively; and
$\sigma = \frac{1}{2}(m_a + m_b + m_c)$.

GROUP I. PART I.

$\Delta =$

1. $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$

2. $\frac{4}{3}\sqrt{\sigma(\sigma - m_a)(\sigma - m_b)(\sigma - m_c)}$
$= \frac{1}{3}\sqrt{2m_a^2m_b^2 + 2m_b^2m_c^2 + 2m_c^2m_a^2 - m_a^4 - m_b^4 - m_c^4}$

3. $1 \div$
$$\sqrt{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}$$
$= h_a^2h_b^2h_c^2 \div$
$$\sqrt{(h_ah_b + h_bh_c + h_ch_a)(-h_ah_b + h_bh_c + h_ch_a)(h_ah_b - h_bh_c + h_ch_a)(h_ah_b + h_bh_c - h_ch_a)}$$

4. $\sqrt{rr_ar_br_c}$

5. $\dfrac{abc}{4R}$

6. $\dfrac{2}{27R}\sqrt{(2m_a^2 + 2m_b^2 - m_c^2)(2m_b^2 + 2m_c^2 - m_a^2)(2m_c^2 + 2m_a^2 - m_b^2)}$

7. $\sqrt{\frac{1}{2}Rh_ah_bh_c}$

8. $\dfrac{r}{2}\sqrt{\dfrac{(r_a + r_b)(r_b + r_c)(r_c + r_a)}{R}}$

9. $\frac{1}{2}\sqrt[3]{abch_ah_bh_c}$

10. $$\frac{abc}{r_a + r_b + r_c - r} = \frac{r^2(r_a + r_b)(r_b + r_c)(r_c + r_a)}{abc} = \left\{\frac{a^2 + b^2 + c^2}{\frac{1}{r^2} + \frac{1}{r_a^2} + \frac{1}{r_b^2} + \frac{1}{r_c^2}}\right\}^{\frac{1}{2}}$$

11. $$\sqrt{\tfrac{1}{3}(m_a^2 + m_b^2 + m_c^2) \div \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}\right)}$$
$$= \frac{1}{3h_ah_bh_c}\sqrt{3(m_a^2 + m_b^2 + m_c^2)(h_a^2h_b^2 + h_b^2h_c^2 + h_c^2h_a^2)}$$

12. $2K^2\dfrac{h_ah_bh_c}{abc} = \frac{1}{2}\dfrac{R}{s}(h_ah_b + h_bh_c + h_ch_a)$

GROUP I. PART I.—*Continued.*

$\Delta =$

13. $\left\{\frac{R^2}{12}(m_a^2+m_b^2+m_c^2)(h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2)\right\}^{\frac{1}{6}}$

14. $\frac{\beta_a\beta_b\beta_c}{s}\cdot\frac{a+b}{2c}\cdot\frac{b+c}{2a}\cdot\frac{c+a}{2b}=\frac{1}{4}\beta_a\beta_b\beta_c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a+b+c}\right)$

15. $\frac{3}{8}\beta_a\beta_b\beta_c\left[\frac{1}{\sqrt{2m_a^2+2m_b^2-m_c^2}}+\frac{1}{\sqrt{2m_b^2+2m_c^2-m_a^2}}+\frac{1}{\sqrt{2m_c^2+2m_a^2-m_b^2}}\right.$

$\left.-\frac{1}{\sqrt{2m_a^2+2m_b^2-m_c^2}+\sqrt{2m_b^2+2m_c^2-m_a^2}+\sqrt{2m_c^2+2m_a^2-m_b^2}}\right]$

16. $\frac{1}{2}\sqrt{\frac{1}{2}\beta_a\beta_b\beta_c\left\{h_a+h_b+h_c-\frac{1}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right\}}$

17. $\frac{1}{2}\sqrt{\beta_a\beta_b\beta_c\left\{\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}-\frac{1}{r}\right\}}$

18. $\sqrt{\frac{1}{8}r\beta_a\beta_b\beta_c\left[(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1\right]}$

19. $\frac{1}{4}\sqrt{\frac{\beta_a\beta_b\beta_c}{R}\cdot\frac{(a+b)(b+c)(c+a)}{a+b+c}}$

20. $\frac{Rr}{\beta_a\beta_b\beta_c}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)$

21. $\frac{h_ah_bh_c}{8abc}(r_a+r_b+r_c-r)^2.$

GROUP I. PART II.

22. $2K^2\sin A\sin B\sin C$
$=\frac{1}{2}K^2(\sin 2A+\sin 2B+\sin 2C)$
$=\frac{2}{3}K^2[\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)]$

23. $\frac{1}{2}R(a\cos A+b\cos B+c\cos C)$
$=R[a\cos C\cos A+b\cos A\cos B+c\cos B\cos C]$

24. $\frac{1}{4}(a^2\cot A+b^2\cot B+c^2\cot C)=\frac{1}{4}\cdot\frac{a^2+b^2+c^2}{\cot A+\cot B+\cot C}$

GROUP I. PART II.—*Continued.*

$$J =$$

25.* $$\frac{1}{3}\frac{m_a^2 + m_b^2 + m_c^2}{\cot A + \cot B + \cot C}$$

26. $$\sqrt{2R_i\vartheta_{ai}\vartheta_{bi}\vartheta_c\,[1 + \cos(A - B) + \cos(B - C) + \cos(C - A)]}$$

$$= 2\sqrt{R_i\vartheta_{ai}\vartheta_{bi}\vartheta_c}\sin(A + \tfrac{1}{2}B)\sin(B + \tfrac{1}{2}C)\sin(C + \tfrac{1}{2}A)$$

27. $$R_i^3\sqrt{h_a h_b h_c}\sin A \sin B \sin C$$

28. $$\tfrac{1}{3}R \sin A \sin B \sin C \left(\frac{ab}{h_c} + \frac{bc}{h_a} + \frac{ca}{h_b}\right)$$

29. $$\frac{h_a + h_b + h_c}{2\left(\dfrac{\cos\frac{1}{2}A}{\vartheta_a} + \dfrac{\cos\frac{1}{2}B}{\vartheta_b} + \dfrac{\cos\frac{1}{2}C}{\vartheta_c}\right)}$$

30. $$\frac{\vartheta_a \sin(C + \frac{1}{2}A) + \vartheta_b \sin(A + \frac{1}{2}B) + \vartheta_c \sin(B + \frac{1}{2}C)}{2\left(\dfrac{\cos\frac{1}{2}A}{\vartheta_a} + \dfrac{\cos\frac{1}{2}B}{\vartheta_b} + \dfrac{\cos\frac{1}{2}C}{\vartheta_c}\right)}$$

31. $$\frac{\vartheta_{ai}\vartheta_{bi}\vartheta_c\sqrt{(m_a^2 - h_a^2)(m_b^2 - h_b^2)(m_c^2 - h_c^2)}}{s(a - b)(b - c)(c - a)}.$$

[TO BE CONTINUED].

DEMONSTRATION OF DESCARTES'S THEOREM AND EULER'S THEOREM.

By PROF. G. B. HALSTED, Austin, Texas.

DESCARTES'S THEOREM.

Cutting by diagonals the faces not triangles into triangles, the whole surface of any polyhedron contains a number of triangular faces four less than double the number of summits.

Proof.

For, joining all the summits by a single closed broken line, this cuts the surface into two skew polygons, each of which contains $S - 2$ triangles, where S is the number of summits.

* *Tidsskrift for Mathematik*, 8° Copenhagen, 1883, fifth series, first year, No. 4, p. 136.

EULER'S THEOREM.

The number of faces and summits in any polyhedron taken together exceeds by two the number of its edges.

Proof.

First Case. If all the faces are triangles; then by Descartes's theorem,

$$F = 2(S - 2).$$

But also $2E = 3F$, since each edge belongs to two faces, and so we get a triangle for every time 3 is contained in $2E$. By adding, we have

$$2E = 2F + 2(S - 2),$$

that is,

$$F + S = E + 2.$$

Second Case. If not, all the faces are triangles. Since to any summit go as many faces as edges, we may replace any polygonal face by a pyramidal summit without changing the equality or inequality relation of $F + S$ to $E + 2$; for such replacement only adds the same number to F as to E and changes one face to a summit. But after all polygonal faces have been so replaced, $F + S = E + 2$ by our first case. Therefore the relation is always equality.

[These theorems are substantially one; the second is also due to Descartes, having been published in 1860 in his *Œuvres Inédites.—W. M. T.*]

PROOF OF A PROPOSITION IN MODERN GEOMETRY.

By Mr. R. D. Bohannan, University of Virginia.

A,B are any two points on the curve which is the locus of the intersection of corresponding rays of two homographic ray-systems. If A,B be made the centres of two ray-systems whose corresponding rays intersect on this curve, these two ray-systems are homographic.

The curve which is the locus of corresponding rays of two homographic ray-systems is of the second degree and passes through the two ray-centres. Being of the second degree, it is fixed by fixing on it five points A,B,C,D,E. Take on it any sixth point F. The three rays BC, BD, BE may be taken arbitrarily to correspond to the three rays AC, AD, AE. Suppose the ray BK corresponding to BF does not intersect AF in F, but in K. Then we have two curves of the second degree, one passing through the six points A,B,C,D,E,F and the other passing through the six points A,B,C,D,E,K. But the curves have five points in common. Thus K,F are coincident. Therefore, etc.

If AC, AD, AE, AF, BC, BD, BE, BF are homographic, the six points A, B,C,D,E,F, no three of them being on a straight line, lie on a curve of the second degree. Thus, by the preceding proposition, the rays drawn from any two of the six points A,B,C,D,E,F to the other four are homographic systems.

Chasles proves these propositions in the inverse order to that here given. See *Traité de Géométrie Supérieure*, Chapter XXV, Arts. 544, 416.

SOLUTIONS OF EXERCISES.

9

IT is assumed that when a gate in a water-pipe is closing the pressure increases uniformly and the discharge decreases uniformly. Investigate an expression for the shortest safe time for closing the gate on the basis of these hypotheses: given the length of the pipe, the velocity of the stream, the working pressure, and the greatest admissible pressure to which the pipe may be exposed.

[*W. M. Thornton.*]

SOLUTION.

Put L for the length of the pipe,

V	velocity of the stream,
Ω	sectional area of the pipe,
g	acceleration due to gravity,
D	heaviness of water,
p	working pressure in the pipe,
P	highest admissible pressure in the pipe,
T	required time in seconds.

The kinetic energy of the stream is

$$\frac{D}{2g} . L\Omega V^2.$$

The velocity and pressure at t seconds after the closing of the valve begins are

$$V_t = V\left[1 - \frac{t}{T}\right], \qquad p_t = p + (P - p)\frac{t}{T};$$

and the work done by the stream in entering this region of high pressure is

$$\varrho \int_0^T P_t V_t \, dt = \varrho V \int_0^T \left(1 - \frac{t}{T}\right)\left[p + (P - p)\frac{t}{T}\right] dt$$

$$= \frac{1}{6}\varrho V (P + 2p)\, T.$$

If we neglect the work done against pipe friction we may equate the kinetic energy to the work done. We get thus for the required time

$$T = \frac{D}{g} \cdot \frac{3LV}{P + 2p};$$

and conversely for the water-ram produced by closing the gate in T seconds

$$P - p = \frac{D}{g} \cdot \frac{3LV}{T} - 3p.$$

[*Wm. M. Thornton.*]

19

SHOW geometrically (without using the calculus) that the asymptote of the hyperbolic spiral, $r = a : \theta$, is parallel to the initial line and distant a above it.

[*L. G. Barbour.*]

SOLUTION.

The Cartesian co-ordinates of a point on the spiral are

$$x = a\frac{\cos\theta}{\theta}, \qquad y = a\frac{\sin\theta}{\theta}.$$

When θ decreases without limit x increases without limit, but y converges to limit a. Hence $y = a$ is asymptotic to the spiral. [*L. G. Barbour.*]

26

DETERMINE the maximum right cone B inscribed in a given right cone A, the vertex of B being at the centre of the base of A. [*O. Root, Jr.*]

SOLUTION.

If the altitude of A be a and that of B be $a - x$, the area a of the base of B will be proportional to x^2, and its volume to $y = x^2(a - x)$; and for the maximum

$$\frac{dy}{dx} = 2ax - 3x^2 = 0;$$

$$\therefore\ x = \frac{2}{3}a, \text{ and } a - x = \frac{1}{3}a.$$

[*T. U. Taylor.*]

28

GIVEN the perpendicular, median, and bisector issuing from one and the same vertex of a plane triangle and terminating in the opposite side, to construct the triangle and determine a formula for its area. [*Marcus Baker.*]

SOLUTION.

1. *Construction of the triangle* This part of the problem is old.

2. *Calculation of the area.* Let p,q,r be the given perpendicular, bisector, and median, and P,Q,R their points of termination. Put l,m,n for QR, RP, PQ respectively and $2x$ for the base. Then the area is px and x is determined from the condition [*Euc.* VI, 3]

$$\frac{x+l}{x-l} = \frac{\sqrt{[p^2+(x+m)^2]}}{\sqrt{[p^2+(x-m)^2]}},$$

whence

$$x^2 = \frac{lr^2 - ml^2}{n};$$

$$\therefore \Delta = p\sqrt{\frac{lr^2 - ml^2}{n}}.$$

Or, as will be found after obvious reductions,

$$\Delta = p\sqrt{\left[(2p^2 - q^2)\sqrt{\frac{r^2 - p^2}{q^2 - p^2}} - (2p^2 - r^2)\right]}.$$

[*W. M. Thornton.*]

29

GIVEN a pair of points A,B, if C,D are such that $OD \cdot OC = OA^2 = OB^2$ and $AOC = AOD$, prove that the pair A,B bears similar relation to the pair C,D. Show, also, the existence of a pair E,F which bears the same relation to each of the pairs A,B and C,D. [*Wm. Woolsey Johnson.*]

SOLUTION.

1. Assume O to be the middle point of AB. As an easy deduction from the conditions given, the triangles COA and AOD are similar; also the triangles COB and BOD.

$$\angle CAD = \angle CAO + \angle OCA = \angle COB = \angle BOD,$$

and

$$\angle CBD = \angle CBO + \angle OCB = \angle COA = \angle AOD;$$

also

$$(OB =)\ AO : CB :: OD : BD$$

and

$$(OA =)\ BO : CA :: OD : AD;$$

$\therefore$ the triangles AOD and CBD are similar;

also the triangles BOD and CAD are similar.

If P be a point in CD such that $\angle BAD = \angle CAP$, since $\angle OBD = \angle ACD$, the triangles BAD and CAP are similar.

Dividing $OB\,(=AO) : OD :: AC : AD :: CB : BD$

by $AP : AD :: CP : BD,$

we have $AC : AP :: CB : CP;$

and, since $\angle OAD = \angle CAP = \angle BCD$, the triangles CAP and BCP are similar, and

$$AP : CP :: CP : BP, \text{ or } AP \times BP = CP^2;$$

similarly it may be shown that $AP \times BP = DP^2$.

2. Let $C'A$ be a line parallel to BD through A, and Q a point such that $\angle BQC = \angle C'AC$ and $BQ : CQ :: BD : CA$; then the triangles BDQ and CAQ will be similar and also the triangles CQB and AQD.

Let FQ be a line bisecting $\angle AQB$ and put $FQ^2 = AQ \times BQ$; continue FQ to the point E such that $QE = FQ$; then will E and F satisfy the conditions required. [*Ormond Stone.*]

31

In the triangle ABC draw AD to the point D in BC; then will

$$AB^2 . DC + AC^2 . DB = BC . AD^2 + BD . DC . BC.$$

[*J. R. Spiegel.*]

SOLUTION.

From C and B draw CE and BF meeting AD at right angles at E and F. Since ED and DF are the projections of BD and DC respectively on AD, we have

$$AB^2 . DC = AD^2 . DC + BD^2 . DC + 2AD . ED . DC,$$
$$AC^2 . BD = AD^2 . BD + CD^2 . BD - 2AD . DF . BD.$$

Adding, we obtain the required relation, since $DC + BD = BC$, and, by similar triangles, $ED . DC = DF . BD$. [*A. V. Lane.*]

EXERCISES.

45

LET O,O' be the centres, r,R the radii of the inscribed and circumscribed circles of a triangle ABC; also let A',B',C' be the feet of the perpendiculars from the vertices on the opposite sides, I their intersection and ρ the radius of the circle inscribed in $A'B'C'$. Prove

$$OI^2 = 2r^2 - 2R\rho,$$
$$O'I^2 = R^2 - 4R\rho.$$ [*Wm. E. Heal.*]

46

FIND an equation that shall express approximately the surface of an egg shell. By means of the variations of not more than three constants in the equation, express the various sizes and shapes of eggs. [*H. A. Newton.*]

47

TWO bulls arriving on opposite banks of a river a rods wide, plunge into the water at the same instant and swim toward each other until they meet. One bull can swim m miles and the other n ($m > n$) miles an hour in still water, and the velocity of the river is v miles an hour. Required the equation to the curve described and the distance swam by each bull, and the location of the point of meeting. [*Artemas Martin.*]

48

IN the angle A a point P is given. Construct a triangle ABC whose base BC shall contain P while the sum of the sides AB and AC equals a given length. [*W. M. Thornton.*]

49

A BEAD is placed on a smooth circular wire ring which receives a given angular velocity in its own plane about a point of its circumference: prove that the initial velocity of the bead is half that of a point of the wire the distance of which from the bead equals that of the bead from the fixed point.

[*William Hoover.*]

50

ABC is an equilateral triangle inscribed in a circle. From any point P within the angle BAC, straight lines PA, PB, PC are drawn. Show, *geometrically*, that $PB + PC - PA$ is a minimum when P is on the circumference of the circle about ABC. [*R. D. Bohannan.*]

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